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## XXIX Cycle - II year presentation

# Incremental stability of Filippov systems

- Incremental stability has been established as a powerful tool to prove convergence in nonlinear dynamical systems. It characterizes asymptotic convergence of trajectories with respect to one another rather than towards some attractor known a priori.
- Popular control applications include tracking and regulation, observer design, coordination, and synchronization.
- A dynamical system $\dot{x} = f(t, x)$ is said to be *incrementally exponentially stable* in a forward invariant set *C* if, given any two solutions $x(t) = \psi(t, t_0, x_0)$ and $y(t) = \psi(t, t_0, y_0)$, there exist constants $K \geq 1$ and $c > 0$ such that

$$|x(t) - y(t)| \leq K\, e^{-c(t-t_0)}\, |x_0 - y_0|.$$

- An effective approach to obtain sufficient conditions for incremental stability comes from contraction theory.
- A *continuously differentiable* vector field $f(x)$ is said to be *contracting* on a convex set *C* if there exists some norm in *C*, with associated matrix measure $\mu$, such that, for some constant $c > 0$ (*contraction rate*)

$$\mu\left(\frac{\partial f}{\partial x}(t, x)\right) \leq -c \quad \forall x \in \mathcal{C}, \;\; \forall t \geq t_0$$

where $\mu(A) := \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}$

- Hence classical contraction analysis requires the system vector field to be continuously differentiable.

- The basic result of nonlinear contraction analysis states that, if a system is contracting, then all of its trajectories are incrementally exponentially stable.
- In this poster we present that our recent work derives conditions to ensure incremental stability of Filippov systems, an important class of discontinuous switched systems.
- Specifically, in our analysis we used results on *regularization* of switched dynamical systems from Sotomayor and Teixeira.
- We then discuss a switching control strategy to either locally or globally incrementally stabilize a class of nonlinear dynamical systems.

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- Switched (or bimodal) Filippov systems are dynamical systems $\dot{x} = f(x)$ where

$$f(x) = \begin{cases} f^+(x) & \text{if } x \in \mathcal{S}^+ \\ f^-(x) & \text{if } x \in \mathcal{S}^- \end{cases}$$

and $f^+, f^- \in C^r(U, \mathbb{R}^n)$, i.e. it is a piecewise continuous vector field having as its discontinuity set a submanifold

$$\Sigma := \{x \in U : H(x) = 0\}$$

called the *switching manifold*, that divides *U* into

$$\mathcal{S}^+ := \{x \in U : H(x) > 0\}$$
$$\mathcal{S}^- := \{x \in U : H(x) < 0\}$$

When the vector fields point both towards $\Sigma$ on either side solutions start *sliding* on it.

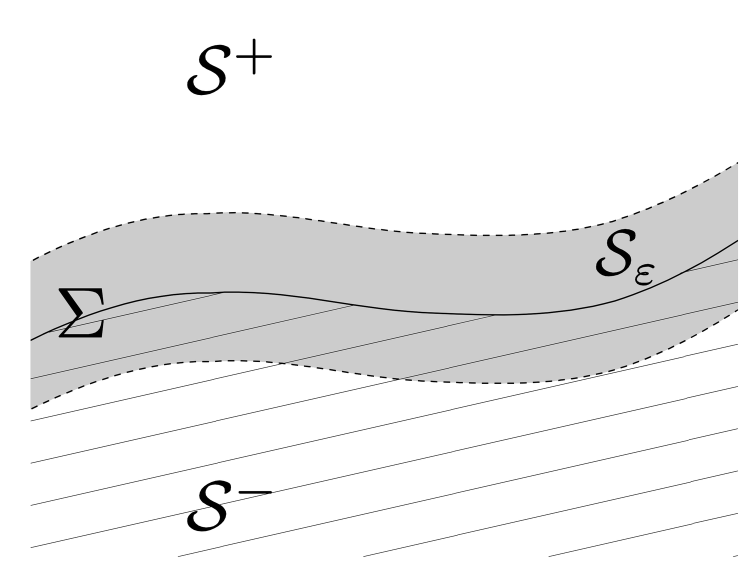
- Filippov systems are of great significance in applications, ranging from problems in electronics (switching elements in power converters), mechanics (friction, impact, constrains), biology and control engineering (relay, sliding mode control, optimal control).
- To study differential properties of discontinuous Filippov vector fields we used a smooth approximation introduced by Sotomayor and Teixeira defined as

$$f_\varepsilon(x) = \frac{1}{2}\left[1 + \varphi\left(\frac{H(x)}{\varepsilon}\right)\right] f^+(x) + \frac{1}{2}\left[1 - \varphi\left(\frac{H(x)}{\varepsilon}\right)\right] f^-(x)$$

where $\varepsilon > 0$ and $\varphi$ is the so-called monotonic *transition function*

$$\varphi(s) = \begin{cases} 1 & \text{for } \;\; s \geq 1, \\ \in (-1, 1) & \text{for } \;\; s \in (-1, 1), \\ -1 & \text{for } \;\; s \leq -1, \end{cases}$$

- Hence the switching manifold $\Sigma$ is replaced with a *boundary-layer* $\mathcal{S}_\varepsilon$




- There exists a *singular perturbation problem* such that fixed points of the boundary-layer model are critical manifolds, on which the motion of the slow variables is described by the reduced problem, which coincides with the sliding equations. Furthermore, denoting with $x_\varepsilon(t)$ a solution of the *regularized* system and with $x(t)$ a solution of the *discontinuous* system with the same initial conditions $x_0$, it holds that

$$|x_\varepsilon(t) - x(t)| = O(\varepsilon), \quad \forall t \geq t_0, \;\; \forall x_0.$$

- From the last relation on regularized systems the following result holds:

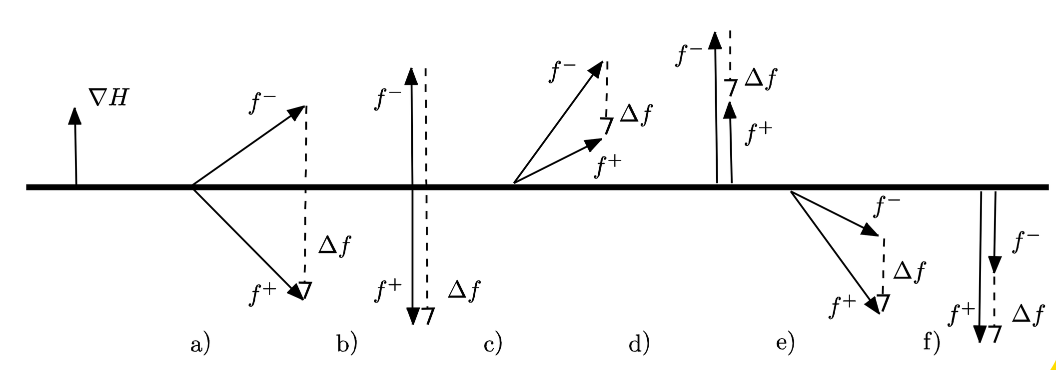
**Theorem**: *If the regularized system $\dot{x} = f_\varepsilon(x)$ is incrementally exponentially stable so it is the Filippov system from which it is derived.*

- Furthermore, sufficient conditions for IES are given from the following theorem obtained taking the limit for $\varepsilon \to 0^+$:

**Theorem**: *A bimodal Filippov systems is IES in a forward-invariant set C if there exists some norm in C, with associated matrix measure $\mu$, such that for some positive constants $c_1$, $c_2$*

$$\mu\left(\frac{\partial f^+}{\partial x}(x)\right) \leq -c_1, \quad \forall x \in \bar{\mathcal{S}}^+$$
$$\mu\left(\frac{\partial f^-}{\partial x}(x)\right) \leq -c_2, \quad \forall x \in \bar{\mathcal{S}}^-$$
$$\mu\Big(\Delta f(x) \cdot \nabla H(x)\Big) = 0, \quad \forall x \in \Sigma.$$

- The first two conditions implies $f^+$ and $f^-$ being contracting. The third condition in Euclidean norm has the following geometrical interpretation in the plane:

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Part of the research activities have been carried out in collaboration with Prof **John Hogan** of the University of Bristol where I spent several research visits over the past year.

Prof John Hogan is Professor of Mathematics since 1992 and he heads both the Bristol Centre for Complexity Sciences and the Bristol Centre for Applied Nonlinear Mathematics.

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- The previous theoretical results has been used to design a switching control strategy to incrementally stabilize a nonlinear system of the form

$$\dot{x} = f(x) + g(x)\, u(x)$$

using a control input

$$u(x) = \begin{cases} u^+(x) & \text{if } H(x) > 0 \\ u^-(x) & \text{if } H(x) < 0 \end{cases}$$

*(to appear in European Control Conference 2016)*

- Other applications are:
  - Use of LMIs to select appropriate metrics
  - Observer design
  - Synchronization of complex networks with discontinuous dynamics